Imaginary Verma modules for the extended affine Lie algebra $sl_2(\mathbb{C}_q)$

M. Dokuchaev, L. M. Vasconcellos Figueiredo
and V. Futorny *

Abstract

We consider one of the most natural extended affine Lie algebras , the algebra $sl_2(\mathbb{C}_q)$, over the quantum torus $\mathbb{C}_q$ and begin a theory of its representations. In particular, we study a class of imaginary Verma modules, obtain for them a criterion of irreducibility and describe their submodule structure when the modules are in "general position".

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*Regular Associate of the ICTP

1 Introduction

In recent years Extended affine Lie algebras (EALA's) have been studied extensively (see [AABGP] and references therein). These algebras provide a natural generalization of affine Kac-Moody Lie algebras and more generally of toroidal Lie algebras. Certain representations of toroidal Lie algebras were considered in [BC], [CF], [FK]. In particular, *imaginary Verma modules* were studied in [FK]. For affine Lie algebras these modules were studied in [F] and in [C]. Some vertex operator representations for toroidal Lie algebras were constructed in [EMY1], [EMY2], [EM], [B], [BB], [T1], [T2]. Also in [BB] certain induced irreducible modules were considered.

Vertex operator representations for extended affine Lie algebras were discussed in [BS], [G1], [G2], [T3], [W], [Y]. A class of extended affine Lie algebras coordinatized by a quantum torus (cf. [M]) contains in particular all toroidal and all affine Lie algebras. Such algebras were considered in [BS], [G1] and [G2].

In the present paper we consider an extended affine Lie algebra $sl_2(\mathbb{C}_q)$ over the *quantum torus* $\mathbb{C}_q$ and study imaginary Verma modules for this algebra. Our results generalize the results in [FK] about the imaginary Verma modules. In particular, we obtain a criterion of irreducibility for such modules and describe the submodule structure and irreducible quotients in the case when at least one central element acts non-trivially.

The structure of the paper is the following. In Section 2, we provide all necessary preliminaries on the quantum torus and our algebra. In Section 3, we study Verma modules for the generalized Heisenberg subalgebra. Theorem 1 describes the irreducible quotient of a Verma module for the generalized Heisenberg subalgebra when a particular central element acts non-trivially. Finally, in Section 4, we study imaginary Verma modules. Here, Corollary 1 establishes a criterion of irreducibility for imaginary Verma modules. Theorem 2 describes the submodule structure of imaginary Verma modules in the case when at least one central element acts non-trivially. Corollaries 2 and 3 give a description of the irreducible quotients in this case.

2 Preliminaries

2.1 The Quantum Torus

Let $q \in M_n(\mathbb{C})$ be a $n \times n$ matrix $q = (q_{ij})$, $1 \leq i, j \leq n$, such that $q_{ii} = 1$ with $q_{ij} = q_{ji}^{-1}$. Let $\mathbb{C}_q$ be the complex algebra spanned by $t_i^{\pm 1}$, $1 \leq i \leq n$ and the product $t_i t_j$ defined by $t_i t_j = q_{ij} t_j t_i$ for all i, j with $1 \leq i, j \leq n$. Note that for all $k, l \in \mathbb{Z}$,

$$t_i^k t_j^l = q_{ij}^{kl} t_j^l t_i^k.$$

Let $\Lambda = \mathbb{Z}^n$ and $a \in \Lambda$, $a = (a_1, a_2, ..., a_n)$ and define $t^a = t_1^{a_1} t_2^{a_2} ... t_n^{a_n}$ and $\mathbb{C}_q^a = \mathbb{C} t^a$. For $a, b \in \Lambda$ we have that

$$t^a t^b = \sigma(a, b) t^{a+b},$$

where $\sigma : \Lambda \times \Lambda \to \mathbb{C}$ is defined by

$$\sigma(a,b) = \prod_{1 \le i < j \le n} q_{ji}^{a_j b_i}.$$

Then

$$\sigma(b,a) = \prod_{1 \le i < j \le n} q_{ji}^{b_j a_i}.$$

and

$$\sigma(b,a)^{-1} = \prod_{1 \le i < j \le n} q_{ij}^{a_i b_j}.$$

Here we use the notation from [BGK]. Define

$$f(a,b) = \sigma(a,b)\sigma(b,a)^{-1} = \prod_{i \ne j} q_{ij}^{a_i b_j}.$$

Note that $[t^a, t^b] = t^a t^b - t^b t^a = \sigma(a,b) t^{a+b} - \sigma(b,a) t^{a+b} = \sigma(b,a)(f(a,b)-1)t^{a+b}$.

The basic properties of f are:

Proposition 1. *For all $a, b, a', b' \in \Lambda$ we have:*

1. $f(a+a', b) = f(a,b) f(a',b)$;
2. $f(a, b+b') = f(a,b) f(a,b')$;
3. $f(b,a) = f(a,b)^{-1}$;
4. $f(a,a) = 1$; $f(a,-a) = 1$;

Define the *radical* of $\mathbb{C}_q$ by

$$R = \{a \in \Lambda \mid f(a,b) = 1 \, \forall b \in \Lambda\}.$$

Remarks:

1. Note that the algebra $\mathbb{C}_q$ is commutative if and only if $q_{ij} = 1$ for all i, j with $1 \le i, j \le n$, or equivalently if $R = \Lambda$.

2. The *center* of $\mathbb{C}_q$ is

$$Z(\mathbb{C}_q) = \sum_{a \in R} \mathbb{C} t^a$$

and

$$\mathbb{C}_q = Z(\mathbb{C}_q) \oplus [\mathbb{C}_q, \mathbb{C}_q].$$

To see this just observe that $t^a \in [\mathbb{C}_q, \mathbb{C}_q]$ if and only if $a \notin R$.

Now let

$$\mathfrak{g} = sl_2(\mathbb{C}_q) = \left\{ \left(\begin{array}{cc} x & y \\ z & w \end{array} \right) \mid x, y, z, w \in \mathbb{C}_q, \, x + w \in [\mathbb{C}_q, \mathbb{C}_q] \right\}.$$

A basis for $sl_2(\mathbb{C}_q)$ is the set

$$\{x_a, y_a, u_a, w_b \,|\, a, b \in \Lambda, b \notin R\}$$

where

$$x_a = \left(\begin{array}{cc} 0 & t^a \\ 0 & 0 \end{array} \right), y_a = \left(\begin{array}{cc} 0 & 0 \\ t^a & 0 \end{array} \right), u_a = \left(\begin{array}{cc} t^a & 0 \\ 0 & -t^a \end{array} \right), w_b = \left(\begin{array}{cc} t^b & 0 \\ 0 & t^b \end{array} \right).$$

We have that $\mathfrak{g}$ is a $\mathbb{C}$ -Lie algebra with brackets $[\,,\,]_{old}$ being defined by $[x, y]_{old} = xy - yx$ for $x, y \in \mathfrak{g}$.

Call, for $a \in \Lambda$,

$$\mathfrak{g}^a = \begin{cases} \text{span}\{x_a, y_a, u_a, w_a\} & \text{if } a \notin R, \\ \text{span}\{x_a, y_a, u_a\} & \text{if } a \in R. \end{cases}$$

For $i = 1, 2, .., n$, let d_i be the *derivations* on $\mathfrak{g}$ defined by

$$d_i x = a_i x, x \in \mathfrak{g}^a, a = (a_1, a_2, ..., a_n) \in \Lambda.$$

Define $\epsilon : \mathbb{C}_q \to \mathbb{C}$ by

$$\epsilon(t^a) = \begin{cases} 0 & \text{if } \; a \neq (0, 0, ..., 0), \\ 1 & \text{if } \; a = (0, 0, ..., 0) \end{cases}$$

and define an invariant non-degenerate bilinear form $(\,,\,)$ on $\mathfrak{g}$ by $(x, x') = \epsilon(tr(xx'))$ where $tr(\,)$ indicates the usual trace function.

Let $C = \bigoplus_{i=1}^n \mathbb{C}c_i$, $D = \bigoplus_{i=1}^n \mathbb{C}d_i$ and let $\mathfrak{q} = \mathfrak{g} \oplus C \oplus D$ be the Lie algebra with brackets

$$[x, x'] := [x, x']_{new} = [x, x']_{old} + \sum_{i=1}^n ([d_i, x], x')c_i$$

for $x, x' \in \mathfrak{g}$.

$$[c_i, x] = 0 \forall x \in \mathfrak{q}, \, \forall i = 1, 2, ..., n$$

and

$$[D, D] = 0, [d_i, x] = d_i x, \, \forall x \in \mathfrak{g}\,, \forall i = 1, 2, ..., n.$$

Then $\mathfrak{q}$ is a Lie algebra, and it is the *quantum torus* .

The brackets of the basis elements of $\mathfrak{q}$ are:

$$[x_a, y_b] = \frac{\sigma(b, a)(f(a, b) + 1)}{2} u_{a+b} + \frac{\sigma(b, a)(f(a, b) - 1)}{2} w_{a+b} + \delta_{a,-b}\sigma(a, b) \sum_{i=1}^n a_i c_i$$

$\forall a,b\in\Lambda$, and where $\delta_{a,-b}=0$ if $a\neq -b$ and $\delta_{a,-a}=1$,

$$[x_a,w_b]=\sigma(b,a)(f(a,b)-1)x_{a+b}\ \forall a,b\in\Lambda, b\notin R$$
$$[x_a,u_b]=-\sigma(b,a)(f(a,b)+1)x_{a+b}\ \forall a,b\in\Lambda$$
$$[y_a,w_b]=\sigma(b,a)(f(a,b)-1)y_{a+b}\ \forall a,b\in\Lambda, b\notin R$$

$$[y_a,u_b]=\sigma(b,a)(f(a,b)+1)y_{a+b}\ \forall a,b\in\Lambda$$
$$[w_a,w_b]=[u_a,u_b]=\begin{cases}2\sigma(a,-a)\sum_{i=1}^n a_ic_i & \text{if } b=-a\\ 0 & \text{if } a+b\in R, a+b\neq 0\\ \sigma(b,a)(f(a,b)-1)w_{a+b} & \text{if } a+b\notin R\end{cases}$$
$$[u_a,w_b]=\begin{cases}0 & \text{if } a+b\in R\\ \sigma(b,a)(f(a,b)-1)u_{a+b} & \text{if } a+b\notin R.\end{cases}$$

2.2 Roots

Let $\mathfrak{h}=\mathbb{C}h\oplus C\oplus D$ where $h=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Then $\mathfrak{h}$ is a Cartan Subalgebra of $\mathfrak{q}$.

We shall describe the root space decomposition of $\mathfrak{q}$ with respect to $\mathfrak{h}$.

Let $\alpha\in\mathfrak{h}^*$ be such that $\alpha(h)=2$ and $\alpha|_{C,D}=0$. For all $i,j=1,2,...,n$ let $\delta_i\in\mathfrak{h}^*$ be defined by $\delta_i(c_j)=0$, $\delta_i(h)=0$, $\delta_i(d_j)=\delta_{i,j}$ for all $j=1,2,...,n$. We identify $a=(a_1,a_2,...,a_n)\in\Lambda$ with the element $\sum_{i=1}^n a_i\delta_i\in\mathfrak{h}^*$ so that for all $x\in\mathfrak{g}^a$

$$[d_i,x]=a_ix=(\sum_{i=1}^n a_i\delta_i)(d_i)x=a(d_i)x$$

for all $i=1,2,....n$. Then the *root system* Δ of $\mathfrak{q}$ is

$$\Delta=\{\pm\alpha+\Lambda\}\cup\Lambda\backslash\{0\}.$$

The *root space decomposition* of $\mathfrak{q}$ with respect to $\mathfrak{h}$ is

$$\mathfrak{q}=\mathfrak{h}\oplus\sum_{\beta\in\Delta}\mathfrak{q}_\beta=\sum_{\beta\in\Delta\cup\{0\}}\mathfrak{q}_\beta,$$

where

$$\mathfrak{q}_\beta=\mathbb{C}x_a$$

if $\beta=\alpha+a, a\in\Lambda$

$$\mathfrak{q}_\beta=\mathbb{C}y_a$$

if $\beta=-\alpha+a$ if $a\in\Lambda$

$$\mathfrak{q}_\beta=\mathbb{C}u_a\oplus\mathbb{C}w_a$$

if $\beta=a\,, a\notin R$

$$\mathfrak{q}_\beta=\mathbb{C}u_a$$

if $\beta=a, a\in R$.

3 Heisenberg subalgebra

Let $\mathfrak{t}$ be the *generalized Heisenberg subalgebra* of $\mathfrak{q}$,

$$\mathfrak{t} = \sum_{a \in \Lambda \setminus \{0\}} \mathfrak{q}_a \oplus C.$$

We can order the elements of Λ lexicographically, that is, for $a, b \in \Lambda, a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$, $a < b$ if and only if, for some $i = 1, ..., n$, $a_i < b_i$, and for all $j > i$, $a_j = b_j$. Set $\Lambda^- = \{a \in \Lambda | a < 0 = (0, \ldots, 0)\}$.

Consider the following Lie subalgebras of $\mathfrak{t}$,

$$\mathfrak{t}^{\pm} = \sum_{\substack{a>0 \\ (a<0)}} \mathfrak{q}_a$$

and

$$\mathfrak{b} = C \oplus \mathfrak{t}^+,$$

a *Borel subalgebra* of $\mathfrak{t}$.

Fix $\lambda \in C^*$. Let $H(\lambda)$ be the *Verma-type module for* $\mathfrak{t}$ *associated to* λ, that is

$$H(\lambda) = U(\mathfrak{t}) / < \mathfrak{t}^+, c_i - \lambda(c_i)1, i = 1, ..., n >,$$

where $U(\ )$ denotes the universal enveloping algebra. From the general theory of Verma modules for Lie algebras we know that if $V = \mathbb{C}v_\lambda$ is a one-dimensional vector space and if we make V a $\mathfrak{b}$- module as $\mathfrak{t}^+ v_\lambda = 0$ and for $i = 1, 2, ..., n$, $c_i v_\lambda = \lambda(c_i) v_\lambda$, then

$$H(\lambda) \cong U(\mathfrak{t}) \otimes_{U(\mathfrak{b})} \mathbb{C}v_\lambda.$$

Because $U(\mathfrak{t}) = U(\mathfrak{t}^-) \otimes U(C) \otimes U(\mathfrak{t}^+)$ we have that

$$H(\lambda) \cong U(\mathfrak{t}^-) \otimes_{\mathbb{C}} \mathbb{C}v_\lambda$$

as vector spaces.

We remark that the elements $d_i, i = 1, ..., n$ of $\mathfrak{q}$ act on $H(\lambda)$ as

$$d_i . v_\lambda = 0$$

and if $r \geq 1$ and if for $j = 1, ...r, z_{a_j} \in \{u_{a_j}, w_{a_j}\}$,

$$d_i . (z_{a_1} ... z_{a_r} v_\lambda) = \sum_{j=1}^{r} z_{a_1} z_{a_2} ... [d_i, z_{a_j}] ... z_{a_r} v_\lambda,$$

defining a Λ-gradation on $H(\lambda)$

$$H(\lambda) = \sum_{\beta \in \Lambda} H(\lambda)_\beta,$$

where, if $\beta = (\beta_1, ..., \beta_n)$,

$$H(\lambda)_\beta = \{x \in H(\lambda) | d_i.x = \beta_i x \,\forall i = 1, ..., n\}.$$

If $x \in H(\lambda)_\beta$ we say that x is *homogeneous of degree* $\beta \in \Lambda$.

Our goal is to describe the irreducible quotients of $H(\lambda)$. For $a = (a_1, a_2, ..., a_n) \in \Lambda$ define

$$\mathfrak{t}_{(a)} = \sum_{l \in \mathbb{Z} \setminus \{0\}} \mathfrak{q}_{la} \oplus \mathbb{C}(a_1 c_1 + a_2 c_2 + ... + a_n c_n).$$

Then $\mathfrak{t}_{(a)}$ is a Heisenberg subalgebra of $\mathfrak{t}$ as in the affine case. Consider the $\mathfrak{t}_{(a)}$-submodule $H_{(a)}$ of $H(\lambda)$ generated by v_λ. Form

$$\hat{H}_{(a)} = \sum_{\beta \in \Lambda \setminus \{(0,...,0)\}} (H_{(a)})_\beta.$$

Let $\lambda \in C^*$ and let

$$\Lambda_\lambda = \{(r_1, r_2, ..., r_n) \in \Lambda \,|\, r_1\lambda(c_1) + r_2\lambda(c_2) + ... + r_n\lambda(c_n) = 0, (r_1, r_2, ..., r_n) \neq (0, 0, ..., 0)\}.$$

For each r with $1 \leq r \leq n$ define

$$R_r = \{a = (a_1, ..., a_r, 0, ..., 0) \in \Lambda \,|\, f(a, b) = 1 \,\text{for all}\, b = (b_1, ..., b_r, 0, ..., 0) \in \Lambda\}.$$

Proposition 2. *Let $a \in \Lambda_\lambda \cap R_r$. The $\mathfrak{t}$-submodule $\tilde{H}_{(a)} = U(\mathfrak{t})\hat{H}_{(a)}$ is a proper submodule of $H(\lambda)$.*

Proof. Assume in contradiction that $\tilde{H}_{(a)} = H(\lambda)$. Then $v_\lambda \in \tilde{H}_{(a)}$ and there exist a monomial $x \in \hat{H}_{(a)}$ and a monomial $y \in U(\mathfrak{t})$ such that v_λ appears in the support of yx. In particular, $yx \neq 0$. We can write $x = x_{l_1 a} x_{l_2 a} ... x_{l_m a} v_\lambda$ where each $l_i a < 0$ and $x_{l_i a} \in \{u_{l_i a}, w_{l_i a}\}$. Then $x \in (\hat{H}_{(a)})_\beta$ with $\beta = \sum_{i=1}^m l_i a$. Let y be of degree γ. If $\gamma < -\beta$ then $\gamma + \beta < 0$ and $yx \in (\hat{H}_{(a)})_{\gamma+\beta} \not\ni v_\lambda$, which is not possible. Hence $\gamma \geq -\beta > 0$. Write, without loss of generality, $y = z_b z_{b'} ... z_{\tilde{b}}$ where $b \leq b' \leq ... \leq \tilde{b}$ and $z \in \{u, w\}$. Then $\gamma = b + b' + ... + \tilde{b}$ and clearly, $\tilde{b} > 0$. If $\tilde{b} = (\tilde{b}_1, ..., \tilde{b}_s, 0, ..., 0)$ with $\tilde{b}_s \neq 0$ and $s > r$ then $\tilde{b} + l_i a > 0$ for all i and $z_{\tilde{b}} x = 0$ which implies that $yx = 0$. So $s \leq r$.

If $\tilde{b}$ is different from each $-l_i a (i = 1, ..., m)$, then $[z_{\tilde{b}}, x_{l_i a}] = 0$ as $-l_i a \in R_r, (i = 1, ..., m)$. It follows that $z_{\tilde{b}} x = z_{\tilde{b}} x_{l_1 a} ... x_{l_m a} v_\lambda = x_{l_1 a} ... x_{l_m a} z_{\tilde{b}} v_\lambda = 0$, and thus $yx = 0$, a contradiction. Hence we may assume that $\tilde{b} = -l_i a$ for a fixed i. Then

$$[z_{\tilde{b}}, x_{l_i}] v_\lambda = [z_{-l_i a}, x_{l_i a}] v_\lambda = -2\sigma(-l_i a, l_i a) l_i \sum_{j=1}^n a_j \lambda(c_j) v_\lambda = 0$$

as $l_i a \in \Lambda_\lambda$. Moreover $[z_{\tilde{b}}, x_{l_j a}] = 0$ for all j with $l_j \neq l_i$, because $l_i a \in R_r$. Then again we obtain that $z_{\tilde{b}} x = 0$, the final contradiction.

$\square$

Set

$$\tilde{H} = \sum_{a \in \Lambda_\lambda \cap (\cup_{r=1}^n R_r)} \tilde{H}_{(a)}.$$

It follows from Proposition 2 that $\tilde{H}$ is a proper $\mathfrak{t}$ - submodule of $H(\lambda)$. Next is the main result in this section.

Theorem 1. *Let $\lambda \in C^*$. If $\lambda(c_1) \neq 0$ then $H(\lambda)/\tilde{H}$ is irreducible.*

In order to prove the theorem we shall need several lemmas.

Lemma 1. *Define an ordering $\preceq$ on Λ^- by:*

1. *$(-k, 0, ..., 0) \prec (0, ..., 0)$ if $k > 0$; $(-k, 0, ..., 0) \prec (-m, 0, ..., 0)$ with $k, m > 0$ if $k < m$.*

   For $r > 1$,

2. *$(k_1, ...k_{r-1}, -k, 0, ..., 0) \prec (m_1, ..., m_{r-1}, 0, ..., 0)$ if $k > 0$.*

3. *$(k_1, ..., k_{r-1}, -k, 0, ..., 0) \prec (m_1, ..., m_{r-1}, -m, 0, ..., 0)$ with $k, m > 0$ if either $k < m$ or $k = m$ and $(k_1, ..., k_{r-1}, 0, ..., 0) < (m_1, ..., m_{r-1}, 0, ..., 0)$ in the previous ordering.*

Define an ordering on the basis $\{w_a, u_a, a \in \Lambda^-\}$ of $\mathfrak{t}^-$ putting $w_a < u_a$ for $a \in \Lambda^-$ and $z_a < z_b$ if $a \prec b$, where $z_a \in \{u_a, w_a\}, z_b \in \{u_b, w_b\}$.
The monomials

$$z_{a_1} ... z_{a_s} v_\lambda$$

with $a_i \in \Lambda^-$, and $z_{a_1} \leq z_{a_2} \leq ... \leq z_{a_s}$ form a basis of $H(\lambda)$.

Proof. Just apply the P-B-W Theorem. ∎

Lemma 2. *Let $b = (b_1, ..., b_r, 0, ..., 0) \in \Lambda \setminus R_r$. Then for arbitrary positive integers $k_1, ..., k_{r-1}$ there exist integers $N_i > k_i$ $(i = 1, ..., r-1)$ such that*

$$f((-N_1, -N_2, ..., -N_{r-1}, 1, 0, .., 0), (b_1, b_2, ..., b_r, 0, ..., 0)) \neq 1.$$

Proof. Observe first that an element $x = (x_1, ..., x_r, 0, ..., 0) \in \Lambda$ is in R_r if and only if

$$(1) \quad \begin{cases} q_{21}^{x_2} q_{31}^{x_3} q_{41}^{x_4} ... q_{r1}^{x_r} & = 1 \\ q_{21}^{-x_1} q_{32}^{x_3} q_{42}^{x_4} ... q_{r2}^{x_r} & = 1 \\ q_{31}^{-x_1} q_{32}^{-x_2} q_{43}^{x_4} ... q_{r3}^{x_r} & = 1 \\ \dots\dots\dots\dots\dots\dots & \\ q_{r1}^{-x_1} q_{r2}^{-x_2} q_{r3}^{-x_3} ... q_{rr-1}^{-x_{r-1}} & = 1. \end{cases}$$

Indeed, $x \in R_r$ if and only if $f(x, b) = 1$ for all $b = (b_1, ..., b_r, 0, ..., 0) \in \Lambda$. It is easy to see that

$$1 = f(x, b)$$

$$= (q_{21}^{x_2}q_{31}^{x_3}q_{41}^{x_4}....q_{r1}^{x_r})^{b_1}(q_{21}^{-x_1}q_{32}^{x_3}q_{42}^{x_4}....q_{r2}^{x_r})^{b_2}(q_{31}^{-x_1}q_{32}^{-x_2}q_{43}^{x_4}....q_{r3}^{x_r})^{b_3}...(q_{r1}^{-x_1}q_{r2}^{-x_2}q_{r3}^{-x_3}....q_{rr-1}^{-x_{r-1}})^{b_r}.$$

Since this holds for all $b_1, ..., b_r$, it follows that $x \in R_r$ if and only if x satisfies (1). Now let $N_1, ..., N_{r-1}$ be positive integers with $N_i > k_i (i = 1, ..., r-1)$. Then the equality $f((-N_1, ..., -N_{r-1}, 1, 0, ..., 0), (b_1, ..., b_r, 0, ..., 0)) = 1$ implies

$$(q_{21}^{b_2}q_{31}^{b_3}q_{41}^{b_4}....q_{r1}^{b_r})^{N_1}(q_{21}^{-b_1}q_{32}^{b_3}q_{42}^{b_4}....q_{r2}^{b_r})^{N_2}(q_{31}^{-b_1}q_{32}^{-b_2}q_{43}^{b_4}....q_{r3}^{b_r})^{N_3}...(q_{r-11}^{-b_1}q_{r-12}^{-b_2}q_{r-13}^{-b_3}....q_{rr-1}^{b_r})^{N_{r-1}}$$

$$= q_{r1}^{-b_1}q_{r2}^{-b_2}q_{r3}^{-b_3}...q_{rr-1}^{-b_{r-1}}.$$

If this holds for all $N_1, ..., N_{r-1}$ with $N_i > k_i$ then each factor in brackets has to be 1, which gives (1). So $b \in R_r$, a contradiction which proves the lemma.

□

Lemma 3. *Let $0 \neq x \in H(\lambda)$ be homogeneous of degree $\alpha \in \Lambda$, $\alpha = (\tilde{\alpha}, -L, 0, ..., 0)$, $L > 0$, $\tilde{\alpha} \in \mathbb{Z}^{r-1}$. Suppose that at least one monomial of x contains a factor $z_{a'}$ with $a' = (a'_1, ..., a'_r, 0, ..., 0) \notin R_r$ and $a'_r \neq 0$. If $x \notin \tilde{H}$ then there exists $y \in \mathfrak{t}$ such that $yx \notin \tilde{H}$ and $yx \in H(\lambda)_\beta$ with $\beta = (\tilde{\beta}, -L+1, 0, ..., 0)$.*

Proof. We assume that all monomials $M = z_{a_1}z_{a_2}...z_{a_s}v_\lambda$ ($z \in \{u, w\}, a_i \in \Lambda^-$), are ordered as in Lemma 1. If some $a_i \in R_r$, then $f(a_i, a_j) = 1$ for all $j = 1, ..., s$, so that we can write $M = z_{a_{i_1}}....z_{a_{i_t}}z'v_\lambda$ where $z' = z_{a_{j_1}}...z_{a_{j_{s-t}}}$ with $a_{j_l} \in R_r$. We say that such monomial has length $t+1$ (so the length of $z'v_\lambda$ is 1). Take an arbitrary ordering on the set of all monomials $z'v_\lambda$ and, using it, order the set of monomials $z_{a_1}z_{a_2}...z_{a_s}z'v_\lambda$ of a fixed length $s+1$ lexicographically from left to right. Write

$$x = \sum_{i \in I} \lambda_i z_i v_\lambda$$

with

$$z_i = z_{a_{1i}}^{t_{1i}} z_{a_{2i}}^{t_{2i}} ... z_{a_{s_i i}}^{t_{s_i i}} z'_i,$$

where each $t_{li} \geq 1$ and $a_{j_1 i} \neq a_{j_2 i}$ if $j_1 \neq j_2$. Among all monomials $z_i v_\lambda$ consider those which have maximal length and denote the corresponding subset of indices by I'. Let $z_{i_0}v_\lambda$ be the smallest monomial in $X' = \{z_i v_\lambda \,|\, i \in I'\}$ and set $X = \{z_i v_\lambda \,|\, i \in I\}$.

For each $i \in I$ and $1 \leq l \leq s_i$, write $a_{li} = (\tilde{a}_{li}, -m_{li}, 0, ..., 0)$, where $m_{li} \geq 0$, $\tilde{a}_{li} = (k_{li}^{(1)}, k_{li}^{(2)}, ..., k_{li}^{(r-1)})$ and set $k_j = \max_{i \in I, 1 \leq l \leq s_i}\{|k_{li}^{(j)}|\}$. We consider separately the cases $m_{1i_0} > 1$ and $m_{1i_0} = 1$. So suppose first that $m_{1i_0} > 1$. By Lemma 2, there exist $N_1, ..., N_{r-1}$ with $N_j > k_j$ such that

$$f((-N_1, ..., -N_{n-1}, 1, 0, ..., 0), (\tilde{a}_{1i_0}, -m_{1i_0}, 0, ..., 0)) \neq 1,$$

so setting $\bar{b} = (-N_1, ..., -N_{r-1})$ and $b = (\bar{b}, 1, 0, ..., 0)$ we have that $f(b, a_{1i_0}) \neq 1$. In particular, $b + a_{1i_0} \notin R_r$ as $f(b + a_{1i_0}, a_{1i_0}) = f(b, a_{1i_0})$. Taking $y = w_b$ we will show that $0 \neq yx \notin \tilde{H}$. First we calculate yz_iv_λ for arbitrary $i \in I$. Set $\xi_{ji} = \sigma(a_{ji}, b)(f(b, a_{ji}) - 1)$.

Because of the choice of b, $a_{ji} \neq -b$ for all $i \in I$, $j = 1, 2, ..., s_i$. Hence $[w_b, z_{a_{ji}}] = \xi_{ji} z_{b+a_{ji}}$. We have

$$yz_i v_\lambda = \xi_{1i} t_{1i} z_{(\tilde{a}_{1i}+\bar{b}, -m_{1i}+1)} z^{t_{1i}-1}_{(\tilde{a}_{1i}, -m_{1i})} z^{t_{2i}}_{a_{2i}} ... z^{t_{s_i i}}_{a_{s_i i}} z'_i v_\lambda$$

$$+ \sum_{j=2}^{s_i} \xi_{ji} t_{ji} z^{t_{1i}}_{(\tilde{a}_{1i}, -m_{1i}, 0, ..., 0)} z^{t_{2i}}_{a_{2i}} ... z_{a_{ji}+b} z^{t_{ji}-1}_{a_{ji}} z^{t_{j+1} i}_{a_{j+1} i} ... z^{t_{s_i i}}_{a_{s_i i}} z'_i v_\lambda$$

$$+ \sum (\text{shorter monomials}). \tag{1}$$

Since $m_{1i} \geq m_{1i_0} > 1$, we see that the first monomial

$$z_{(\tilde{a}_{1i}+\bar{b}, -m_{1i}+1, 0, ..., 0)} z^{t_{1i}-1}_{(\tilde{a}_{1i}, -m_{1i}, 0, ..., 0)} z^{t_{2i}}_{a_{2i}} ... z^{t_{s_i i}}_{a_{s_i i}} z'_i v_\lambda \tag{2}$$

is still ordered in the ordering defined in Lemma 1. Taking a monomial M from (1), not equal to (2) observe that its smallest term is greater than $z_{(\tilde{a}_{1i}+\bar{b}, -m_{1i}+1, 0, ..., 0)}$, because for $j > 1$, either $m_{ji} > m_{1i}$ or $m_{ji} = m_{1i}$ but $\tilde{a}_{ji} > \tilde{a}_{1i}$. The monomial M may not be ordered. If it is not, then after ordering M we obtain the ordered permutation of M which is bigger than (2) and some other shorter summands. Thus we can write

$$yz_i v_\lambda = \xi_{1i} t_{1i} z_{(\tilde{a}_{1i}+\bar{b}, -m_{1i}+1, 0, .., 0)} z^{t_{1i}-1}_{(\tilde{a}_{1i}, -m_{1i}, 0, ..., 0)} z^{t_{2i}}_{a_{2i}} ... z^{t_{s_i i}}_{a_{s_i i}} u'_i v_\lambda +$$

$$\sum (\text{greater ordered monomials of the same length}) + \sum (\text{shorter ordered monomials}), \tag{3}$$

where the first monomial coincides with (2). Let M_0 be the monomial (2) with $i = i_0$. So M_0 is the first monomial in (3) when taking $i = i_0$. Observe that $\xi_{1i_0} \neq 0$ as $f(b, a_{1i_0}) \neq 1$. It is easy to see that M_0 does not cancel out in

$$yx = \sum_{i \in I} \lambda_i w_b z_i v_\lambda.$$

Indeed, it follows from (3) that it is enough to check that M does not coincide with (2) when $i \neq i_0$. But this obviously follows from the choice of i_0.

Observe that $\tilde{a}_{1i_0} + b < 0$ and that M_0 has degree $\beta = (\tilde{\alpha} + \bar{b}, -L + 1, 0, ..., 0)$. Notice also that M_0 does not belong to $\tilde{H}$. This is because $(\tilde{a}_{1i_0} + \bar{b}, -m_{1i_0} + 1, 0, ..., 0) \notin R_r$ and in view of $x \notin \tilde{H}$, no $a_{ji_0} (j = 2, ..., s_{i_0})$ can be a multiple of some $a \in \Lambda_\lambda \cap R_r$.

Now consider the case when $m_{1i_0} = 1$. Suppose first that there exists l, $1 \leq l \leq s_{i_0}$, such that $m_{li_0} > 1$. Let l' be the least such index. By Lemma 2, there exist $N_1, ..., N_{r-1}$ with $N_j > k_j$ such that

$$f((-N_1, ..., -N_{n-1}, 1, 0, ..., 0), (\tilde{a}_{l' i_0}, -m_{l' i_0}, 0, ..., 0)) \neq 1.$$

As before , we take $\bar{b}=(-N_1,...,-N_{r-1})$, $b=(\bar{b},1,0,...,0)$, $y=w_b$ and set $\xi_{ji}=\sigma(a_{ji},b)(f(b,a_{ji})-1)$. Then $\xi_{l'i_0}\neq 0$ and $[w_b,z_{a_{ji}}]=\xi_{ji}z_{b+a_{ji}}$. Let $i\in I'$ be an index with similar property: $m_{l''i}>1$ for some $1\leq l''\leq s_i$. Suppose that l'' is the least such index. We have that

$$yz_iv_\lambda=\xi_{l''i}t_{l''i}z_{a_{1i}}^{t_{1i}}...z_{a_{l''i}+b}z_{a_{l''i}}^{t_{l''i}-1}...z_{a_{s_ii}}^{t_{s_ii}}z_i'v_\lambda+$$

$$\sum_{j\neq l''}\xi_{ji}t_{ji}z_{a_{1i}}^{t_{1i}}...z_{a_{ji}+b}z_{a_{ji}}^{t_{ji}-1}...z_{a_{s_ii}}^{t_{s_ii}}z_i'v_\lambda+$$

$$\sum(\text{shorter monomials}). \tag{4}$$

We may still need to reorder these monomials, but after doing so we will get only shorter summands. A routine check shows that among the monomials of maximal length in (4) the smallest one is the first summand. Let M_0 be the first monomial of (4) when taking $i=i_0$ then (l'' becomes l'). We claim that M_0 does not cancel out in yx. We see that it can not happen in yz_iv_λ for every $i\in I'$ with the above property of existence of l''. Now, if $i\in I'$ does not have such a property, then the last entries of all a's involved in z_i are all -1's and 0's. Thus when acting on z_iv_λ by $y=w_{(\bar{b},1,0,...,0)}$we reduce the number of -1's, while in M_0 it is either the same as in z_{i_0} (case $m_{l'i_0}>2$), or even greater (case $m_{l'i_0}=2$). Hence, M_0 does not cancel out in such yz_iv_λ. Finally, if $i\notin I'$, then all monomials of yz_iv_λ which appear after reordering are shorter than M_0. Thus, M_0 belongs to the support of yx, as $\xi_{l'i_0}\neq 0$. Since $M_0\notin\tilde{H}$, $0\neq yx\neq\tilde{H}$. Moreover, the degree of M_0 is $\beta=(\tilde{\alpha}+\bar{b},-L+1,0,...,0)$.

Suppose now that $m_{1i_0}=1$ and $m_{li_0}\in\{0,1\}$ for all $1\leq l\leq s_{i_0}$. Let $z_{i_1}v_\lambda$ be the smallest monomial in $X'\setminus\{z_{i_0}v_\lambda\}$. If again $m_{1i_1}=1$ and $m_{li_1}\notin\{0,1\}$ for all $1\leq l\leq s_{1i}$, then we take the next smallest monomial and continue until we get the first monomial $z_{i_t}v_\lambda$ with some $m_{li_t}>1$, $1\leq l\leq s_{i_t}$. Let l' be the minimal such index. Let also l'' be the maximal index with $m_{l''i_0}=1$. Set $I''=\{i_0,i_1,...,i_{t-1}\}$. By Lemma 2 we choose $N_1,...,N_{r-1}$ with $N_j>k_j$ such that $f((-N_1,...,-N_{r-1},1,0,...,0),a_{l''i_0})\neq 1$. Take $b=(\bar{b},1,0,...,0), y=w_b$. Then $\xi_{l''i_0}\neq 0$ where ξ_{ji} has the same meaning as before. For $i\in I''$ we let l''' be the maximal index with $m_{l'''i}=1$. We have

$$yz_iv_\lambda=\xi_{l'''i}t_{l'''i}z_{a_{1i}}^{t_{1i}}...z_{a_{l'''i+b}}z_{a_{l'''i}}^{t_{l'''i}-1}...z_{a_{s_ii}}^{t_{s_ii}}z_i'v_\lambda$$

$$+\sum_{j\neq l'''}\xi_{ji}t_{ji}z_{a_{1i}}^{t_{1i}}...z_{a_{ji}+b}z_{a_{ji}}^{t_{ji}-1}...z_{a_{s_ii}}^{t_{s_ii}}z_i'v_\lambda$$

$$\sum(\text{shorter monomials}). \tag{5}$$

Among the monomials of maximal length in (5) the first summand is the smallest one. Now let M_1 be the first monomial of (5) with $i=i_0$ (then l''' becomes l'').

Since $z_{i_0}v_\lambda < z_iv_\lambda$, it follows that M_1 does not cancel out in yz_iv_λ for all $i \in I''$.

If we restrict our attention only to $I' \setminus I''$, then we are in conditions of previous cases. Let $z_{i'_0}v_\lambda$ be the smallest monomial among the monomials z_iv_λ of maximal length with $i \in I' \setminus I''$. In view of the previous considerations let M_0 be the first monomial in (2) or (4) with $i = i'_0$. Now, if $M_1 < M_0$, then M_1 can not be cancelled out in yx and it really belongs to the support of yx as $\xi_{l''i_0} \neq 0$.

Let $M_0 \leq M_1$. Taking larger $N_1, \dots, N_{r-1}$ if necessary we can guarantee by Lemma 2 that $M_0 < M_1$ and $f(b, a_{l'i'_0}) \neq 1$ where, as usually, $b = (-N_1, ..., -N_{r-1}, 1, 0, ..., 0)$. Then $\xi_{l'i'_0} \neq 0$ and since M_0 is the smallest one among all monomials of maximal length which appear in yx, it does not cancel out.

$\square$

Proof of Theorem 1.

Suppose that $\lambda(c_1) \neq 0$. We want to show that $H(\lambda)/\tilde{H}$ is irreducible.

Let $0 \neq W$ be a submodule of $H(\lambda)/\tilde{H}$ and let $0 \neq \bar{x} = x + \tilde{H}$ be an element of W. Since $x \notin \tilde{H}$, we may assume that no monomial in x is in $\tilde{H}$. We also may assume that x is homogeneous of degree $\alpha = (\tilde{\alpha}, -L, 0, ..., 0), \tilde{\alpha} \in \mathbb{Z}^{r-1}$ and $L > 0$. Suppose first that $r = 1$.

Consider the subalgebra $\mathfrak{s}$ of $\mathfrak{t}$ with a $\mathbb{C}$ - basis $\{c_1, w_{(i,0,...,0)}, u_{(j,0,...,0)} | \, i, j \in \mathbb{Z}\}$. Then $\mathfrak{s}$ is a Heisenberg subalgebra of $\mathfrak{t}$ as in the affine setting (note that $f((i, 0, 0, ..., 0), (j, 0, 0, ..., 0)) = 1$ for all $i, j \in \mathbb{Z}$ so that the brackets $[w_{(i,0,...,0)}, w_{(j,0,...,0)}] = [u_{(i,0,...,0)}, u_{(j,0,...,0)}] = 0$ unless $i = -j$ and in this case $[w_{(i,0,...,0)}, w_{(-i,0,...,0)}] = [u_{(i,0,...,0)}, u_{(-i,0,...,0)}] = 2ic_1$ and $[u_{(i,0,...,0)}, w_{(j,0,...,0)}] = 0$ for all $i, j \in \mathbb{Z}$.) Since $r = 1$, $x \in U(\mathfrak{s})v_\lambda$ and because $\lambda(c_1) \neq 0$, we have from the results for affine Lie algebras (see [K]) that $U(\mathfrak{s})v_\lambda$ is irreducible, so that $v_\lambda \in U(\mathfrak{s})v_\lambda \subset U(\mathfrak{t})v_\lambda$. Now, suppose that $r > 1$ and that the theorem is true for $r - 1$. If all of the monomials in x contain only factors z_a for $a \in R_r$, we proceed exactly as in the proof of [FK, Theorem 1] and we obtain $v_\lambda \in W$. If at least one monomial in x contains a factor z_a , with $a = (a_1, ..., a_r, 0, ..., 0) \notin R_r$ and $a_r \neq 0$ we proceed by induction on L. Applying Lemma 3 we obtain $0 \neq z \in W$ so that $z \in H(\lambda)_\beta$ with $\beta = (\tilde{\beta}, -K, 0, ..., 0)$ where $\tilde{\beta} \in \mathbb{Z}^{r-2}$ and $K > 0$ and $z \notin \tilde{H}$. The result then follows by the induction hypothesis.

Remark 1. *We believe that Theorem 1 is valid when $\lambda(c_i) \neq 0$ for some i.*

4 Imaginary Verma modules

In this section we consider certain Verma type modules for the extended affine Lie algebra $sl_2(\mathbb{C}_q)$. The results are similar to the case of toroidal Lie algebras [FK].

Let

$$\Delta_\pm = \{\pm\alpha + \Lambda\} \cup \{a \in \Lambda \,|\, \pm a > 0\},$$

$$\mathfrak{q}_\pm = \sum_{\beta \in \Delta_\pm} \mathfrak{q}_\beta \ , \ \mathfrak{q}_-^{re} = \sum_{\beta \in \Delta_- \setminus \Lambda} \mathfrak{q}_\beta.$$

We have the following Cartan type decomposition:

$$\mathfrak{q} = \mathfrak{q}_- \oplus \mathfrak{h} \oplus \mathfrak{q}_+.$$

We will call a subalgebra $B = \mathfrak{h} \oplus q_+$ a *non-standard Borel subalgebra.*

Let $\lambda \in \mathfrak{h}^*$. A $\mathfrak{q}$-submodule

$$M(\lambda) = U(\mathfrak{q}) \otimes_{U(B)} \mathbb{C}v_\lambda,$$

where $\mathfrak{q}_+ v_\lambda = 0$ and $hv_\lambda = \lambda(h)v_\lambda$, is called an *imaginary Verma module.*

This is a weight (with respect to $\mathfrak{h}$) module with $M(\lambda)_\mu \neq 0$ if and only if $\mu \in \{\lambda - \alpha + \Lambda\} \cup \{\lambda - a \,|\, a \in \Lambda, a > 0\}$.

Moreover, $\dim M(\lambda)_\lambda = 1$ and $\dim M(\lambda)_\mu = \infty$ as long as $M(\lambda)_\mu \neq 0$. The module $M(\lambda)$ has a unique irreducible quotient $L(\lambda)$.

Let $\tilde{v}_\lambda = 1 \otimes v_\lambda$. Consider a subspace $\hat{M}(\lambda) = U(\mathfrak{t})\tilde{v}_\lambda$. Then $\hat{M}(\lambda)$ is a $\mathfrak{t}$-submodule of $M(\lambda)$ and $M(\lambda) \cong H(\lambda)$. We will show that this submodule plays a crucial role in the structure of $M(\lambda)$. It follows from the PBW theorem that

$$M(\lambda) \cong U(\mathfrak{q}_-^{re}) \otimes_{\mathbb{C}} \hat{M}(\lambda)$$

as a vector space.

For a submodule $N \subset M(\lambda)$ denote $\hat{N} = N \cap \hat{M}(\lambda)$.

Proposition 3. *Let $0 \neq N \subset M(\lambda)$. Then $\hat{N} \neq 0$.*

Proof. Let $0 \neq v \in N$. We can assume that v is a weight vector and that

$$v = \sum_i y_{a_{i1}} ... y_{a_{im}} z_i \tilde{v}_\lambda,$$

where $a_{ij} \in \Lambda$, $z_i \in U(\mathfrak{t}^-)$ and all the monomials $y_{a_{i1}} ... y_{a_{im}}$ are linearly independent. Note that m is an invariant of v since v is a weight vector. We will denote $\|v\| = m$ and will prove the statement by induction on m. The base of induction $m = 0$ is trivial. Clearly for appropriate $A \in \Lambda$,

$$N \ni x_A v = \sum_i \sum_{j=1}^n y_{a_{i1}} ... [x_A, y_{a_{ij}}] ... y_{a_{im}} z_i \tilde{v}_\lambda \neq 0$$

and $\|x_A v\| = m - 1$. Hence we can apply our induction hypothesis and conclude that $0 \neq u x_A v \in \hat{N}$ for some $u \in U(\mathfrak{q})$.

□

Corollary 1. *Let $\lambda \in \mathfrak{h}^*$. A $\mathfrak{q}$-module $M(\lambda)$ is irreducible if and only if $\hat{M}(\lambda) = H(\lambda)$ is irreducible as a $\mathfrak{t}$-module.*

The algebra $U(\mathfrak{t})$ has a natural Λ -gradation. If $u \in U(\mathfrak{t})$ is a homogeneous element of degree $(a_1, ..., a_n)$ we denote $|u|_i = |a_i|$. For an arbitrary element $u \in U(\mathfrak{t})$ denote $|u|_i = \max_j |u_j|_i$ where u_j are homogeneous components of u.

Theorem 2. *Let $N \subset M(\lambda)$. If there exists an i such that $\lambda(c_i) \neq 0$ then $N \cong U(\mathfrak{q}_-^{re}) \otimes_{\mathbb{C}} \hat{N}$ as vector space.*

To prove the theorem we need the following

Lemma 4. *Let $N \subset M(\lambda)$, $\lambda(c_k) \neq 0$ and $0 \neq u\tilde{v}_\lambda \in N$ where*

$$u = \sum_{i \in I} z_{A_i} u_i$$

with $z \in \{u, w\}$, $A_i = (a_{i1}, ..., a_{in}) < 0$, $A_i \notin \Lambda_\lambda$, $A_i \neq A_j$ if $i \neq j$ except possibly the case when $z_{A_i} = u_{A_i}$ and $z_{A_j} = w_{A_j}$; $u_i \in U(\mathfrak{t})$ and $|u_i|_j << |a_{ij}|$, for all $i \in I$, $j = k$ and $j = n$. Moreover $|a_{1n}| - 2|u|(n) >> 0$, $|u|(n) = max_k |u_k|_n$. Then $u_i \tilde{v}_\lambda \in N$ for all $i \in I$.

Proof. Since $[u_a, w_{-a}] = 0$ we can assume without loss of generality that $A_i \neq A_j$ for all $i \neq j$. Use induction on I. Let $|I| = 1$. Then $u = z_A u'$, $A = (a_1, ..., a_n)$. Apply z_{-A}. Since $z_{-A} u' \tilde{v}_\lambda = 0$ and $A \notin \Lambda_\lambda$ we conclude that $u' \tilde{v}_\lambda \in N$.

Suppose now that $|I| > 1, |a_{1n}| \leq |a_{2n}| \leq ...$ and $(a_{11}, ..., a_{1n-1}) > (a_{i1}, ..., a_{in-1})$ if $a_{in} = a_{1n}$. We can also assume that $f(A_1, A_i) \neq 1$ for some $i \neq 1$ since otherwise application of z_{-A_1} completes the proof. Consider such i. Assume that $k \neq n$ and consider

$$B_{D,M} = (-a_{11}, ..., -a_{1k-1}, -a_{1k} - D, -a_{1k+1}, ..., -a_{1n-1}, -a_{1n} - M),$$

where $D >> 0$ and $|u|(n) << M << |a_{1n}| - |u|(n)$. Suppose that $f(B_{D,M}, A_i) = 1$, for sufficiently large number of pairs (D, M). Then

$$(q_{k1}^{a_{i1}} q_{k2}^{a_{i2}} ... q_{kk-1}^{a_{ik-1}} q_{kk+1}^{a_{ik+1}} ... q_{kn}^{a_{in}})^D (q_{n1}^{a_{i1}} ... q_{nn-1}^{a_{in-1}})^M$$

is a constant for all those pairs (D, M). Therefore

$$q_{k1}^{a_{i1}} q_{k2}^{a_{i2}} ... q_{kn}^{a_{in}} = q_{n1}^{a_{i1}} ... q_{nn-1}^{a_{in-1}} = 1$$

implying that $f(-A_1, A_i) = 1$ which is a contradiction. Thus we can assume that

$$f(B_{D,M}, A_i) \neq 1$$

for a large number of pairs (D, M) and some $i \neq 1$. We can also choose D and M in such a way that $\tilde{A}_i = A_i + B_{D,M} \notin \Lambda_\lambda$ for all $i \in I$. Fix D and M that satisfy conditions above and apply $u_{B_{D,M}}$:

$$v = u_{B_{D,M}} u\tilde{v}_\lambda = [u_{B_{D,M}}, z_{A_1}] u_1 \tilde{v}_\lambda + \sum_{i \neq 1} [u_{B_{D,M}}, z_{A_i}] u_i \tilde{v}_\lambda \in N.$$

If $f(B_{D,M}, A_1) = 1$ then the first term in the sum above is zero and we can complete the proof by induction on $|I|$. Assume now that $f(B_{D,M}, A_1) \neq 1$. Denote by $\tilde{z}_i = [u_{B_{D,M}}, z_{A_i}], \tilde{A}_i = (\tilde{a}_{i1}, ..., \tilde{a}_{in})$. If $f(-\tilde{A}_1, \tilde{A}_i) = 1$ for all $i \neq 1$ then $z_{-\tilde{A}_1}v = [z_{-\tilde{A}_1}, \tilde{z}_1]u_1\tilde{v}_\lambda$, with z here the same as in $\tilde{z}_1$, and therefore $u_1\tilde{v}_\lambda \in N$. Thus we assume that $f(-\tilde{A}_1, \tilde{A}_i) \neq 1$ for some $i \neq 1$. Consider $C_{K,L} = (C_1, ..., C_n)$ where $C_k = D - K >> 0$, $C_n = M - L >> 0$, $C_j = 0$ for $j \neq k, n$. The same argument as above shows that there exists a sufficient number of pairs (K, L) such that $f(C_{K,L}, \tilde{A}_i) \neq 1$ for some $i \neq 1$, and for all such indices i, $C_{K,L} + \tilde{A}_i \notin \Lambda_\lambda$. First note that among all pairs (K, L) there are some for which $f(C_{K,L}, \tilde{A}_1) = 1$. Indeed, $f(C_{K,L}, \tilde{A}_1) = 1$ as soon as $KM = LD$. Without loss of generality we can assume that M and D are both divisible by 2^r for some integer $r >> 0$. Set $K = \frac{1}{2^r}D$, $L = \frac{1}{2^r}M$ and assume that $f(C_{\frac{1}{2^r}D, \frac{1}{2^r}M}, \tilde{A}_i) = 1$. for sufficiently many integers r. Hence

$$q_{k1}^{\tilde{a}_{i1}(D - \frac{1}{2^r}D)} ... q_{kn}^{\tilde{a}_{in}(D - \frac{1}{2^r}D)} q_{n1}^{\tilde{a}_{i1}(M - \frac{1}{2^r}M)} ... q_{nn-1}^{\tilde{a}_{in-1}(M - \frac{1}{2^r}M)} = 1$$

and

$$f(-\tilde{A}_1, \tilde{A}_i) = (q_{k1}^{\tilde{a}_{i1}} ... q_{kn}^{\tilde{a}_{in}})^D (q_{n1}^{\tilde{a}_{i1}} ... q_{nn-1}^{\tilde{a}_{in-1}})^M = 1.$$

We conclude that there exist infinitely many pairs (K, L) satisfying $KM = LD$ and $f(C_{K,L}, \tilde{A}_i) \neq 1$ as long as $f(-\tilde{A}_1, \tilde{A}_i) \neq 1$. Finally, if $C_{\frac{1}{2^r}D, \frac{1}{2^r}M} + \tilde{A}_i \in \Lambda_\lambda$ then

$$0 = ((D - \frac{1}{2^r}) + \tilde{a}_{ik})\lambda(c_k) + (M - \frac{1}{2^r}M)\lambda(c_n) + \sum_{j \neq k,n} \tilde{a}_{ij}\lambda(c_j) =$$

$$(1 - \frac{1}{2^r})(D\lambda(c_k) + M\lambda(c_n)) + \sum_j \tilde{a}_{ij}\lambda(c_j),$$

which is impossible since $\tilde{A}_i \notin \Lambda_\lambda$ for all $i \in I$. Hence, we can choose $0 << K << D$ and $0 << L << M$ such that $f(C_{K,L}, \tilde{A}_1) = 1$ and $f(C_{K,L}, \tilde{A}_i) \neq 1$ for at least one $i \in I$, and in this case $C_{K,L} + \tilde{A}_i \notin \Lambda_\lambda$. Apply $u_{C_{K,L}}$:

$$0 \neq u_{C_{K,L}}v = \sum_{i \neq 1} [u_{C_{K,L}}, \tilde{z}_i]u_i\tilde{v}_\lambda \in N.$$

We complete the proof by induction on $|I|$. The same arguments work in the case $k = n$. The lemma is proved. $\square$

The Theorem 2 is an easy corollary of Lemma 4. The proof follows the general lines of the proof of Theorem 2 in [FK]. We omit the details here.

Corollary 2. *Let $\lambda \in \mathfrak{h}^*$ and $\lambda(c_i) \neq 0$ for some i. Then $L(\lambda) \cong U(\mathfrak{q}_-^{re}) \otimes_{\mathbb{C}} \bar{H}(\lambda)$, where $\bar{H}(\lambda)$ is the maximal submodule of $H(\lambda)$.*

Corollary 3. *Let $\lambda \in \mathfrak{h}^*$ and $\lambda(c_1) \neq 0$. Then $L(\lambda) \cong U(\mathfrak{q}_-^{re}) \otimes_{\mathbb{C}} (H(\lambda)/\tilde{H})$.*

Proof. Follows from Theorem 1 and Theorem 2. □

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Instituto de Matemática e Estatistica,
Universidade de São Paulo
Caixa Postal 66281- CEP 05315-970
São Paulo, Brazil
e-mail: futorny@ime.usp.br